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On The Derivatives and Partial Derivatives of A Certain Generalized Hyper geometric Function

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ABSTRACT
In this paper, methods involving derivatives and the Mellin transformation are employed in obtaining finite summations for the $\overline{H}$ -function of two variables and certain special partial derivatives for the $\overline{H}$ -function of two variables with respect to parameters.

KEY WORDS: Derivatives, Partial derivatives, $\overline{H}$ -function of two variables, Mellin transformation.
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I. INTRODUCTION

The $\overline{H}$ -function of two variables defined and represented by Singh and Mandia [12] in the following manner:

$$\overline{H}[x,y]=\overline{H}\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]=\overline{H}^{o,n_1:\,m_2,n_2:m_3,n_2}_{p_1,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\phi_2(\xi)\phi_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1.1)$$

Where

$$\phi_1(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)} \qquad (1.2)$$

$$\phi_2(\xi)=\frac{\prod_{j=1}^{n_2}\{\Gamma(1-c_j+\gamma_j\xi)\}^{K_j}\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)\prod_{j=m_2+1}^{q_2}\{\Gamma(1-d_j+\delta_j\xi)\}^{L_j}} \qquad (1.3)$$

$$\phi_3(\eta)=\frac{\prod_{j=1}^{n_3}\{\Gamma(1-e_j+E_j\eta)\}^{R_j}\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)\prod_{j=m_3+1}^{q_3}\{\Gamma(1-f_j+F_j\eta)\}^{S_j}} \qquad (1.4)$$

Where x and y are not equal to zero (real or complex), and an empty product is interpreted as unity p_i, q_i, n_i, m_j are non-negative integers such that $0 \leq n_i \leq p_i, o \leq m_j \leq q_j (i = 1, 2, 3;\ j = 2, 3)$. All the $a_j (j = 1, 2, \ldots, p_1), b_j (j = 1, 2, \ldots, q_1), c_j (j = 1, 2, \ldots, p_2), d_j (j = 1, 2, \ldots, q_2),$

$e_j (j = 1, 2, \ldots, p_3),\ f_j (j = 1, 2, \ldots, q_3)$ are complex parameters. $\gamma_j \geq 0 (j = 1, 2, \ldots, p_2), \delta_j \geq 0 (j = 1, 2, \ldots, q_2)$ (not all zero simultaneously), similarly $E_j \geq 0 (j = 1, 2, \ldots, p_3), F_j \geq 0 (j = 1, 2, \ldots, q_3)$ (not all zero simultaneously). The exponents $K_j (j = 1, 2, \ldots, n_3), L_j (j = m_2 + 1, \ldots, q_2), R_j (j = 1, 2, \ldots, n_3), S_j (j = m_3 + 1, \ldots, q_3)$ can take on non-negative values.

The contour L_1 is in ξ -plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma \left(d_j - \delta_j \xi \right) (j = 1, 2, \ldots, m_2)$ lie to the right and the poles of $\Gamma \left\{ \left(1 - c_j + \gamma_j \xi \right) \right\}^{K_j} (j = 1, 2, \ldots, n_2), \Gamma \left(1 - a_j + \alpha_j \xi + A_j \eta \right) (j = 1, 2, \ldots, n_1)$ to the left of the contour. For $K_j (j = 1, 2, \ldots, n_2)$ not an integer, the poles of gamma functions of the numerator in (1.3) are converted to the branch points.

The contour L_2 is in η -plane and runs from $-i\infty$ to $+i\infty$. The poles of $\Gamma \left(f_j - F_j \eta \right) (j = 1, 2, \ldots, m_3)$ lie to the right and the poles of $\Gamma \left\{ \left(1 - e_j + E_j \eta \right) \right\}^{R_j} (j = 1, 2, \ldots, n_3), \Gamma \left(1 - a_j + \alpha_j \xi + A_j \eta \right) (j = 1, 2, \ldots, n_1)$ to the left of the contour. For $R_j (j = 1, 2, \ldots, n_3)$ not an integer, the poles of gamma functions of the numerator in (1.4) are converted to the branch points.

The functions defined in (1.1) is an analytic function of x and y , if

$$U = \sum_{j=1}^{p_1} \alpha_j + \sum_{j=1}^{p_2} \gamma_j - \sum_{j=1}^{q_1} \beta_j - \sum_{j=1}^{q_2} \delta_j < 0 \tag{1.5}$$

$$V = \sum_{j=1}^{p_1} A_j + \sum_{j=1}^{p_3} E_j - \sum_{j=1}^{q_1} B_j - \sum_{j=1}^{q_3} F_j < 0 \tag{1.6}$$

The integral in (1.1) converges under the following set of conditions:

$$\Omega = \sum_{j=1}^{n_1} \alpha_j - \sum_{j=n_1+1}^{p_1} \alpha_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j L_j + \sum_{j=1}^{n_2} \gamma_j K_j - \sum_{j=n_2+1}^{p_2} \gamma_j - \sum_{j=1}^{q_1} \beta_j > 0 \tag{1.7}$$

$$\Lambda = \sum_{j=1}^{n_1} A_j - \sum_{j=n_1+1}^{p_1} A_j + \sum_{j=1}^{m_2} F_j - \sum_{j=m_2+1}^{q_2} F_j S_j + \sum_{j=1}^{n_3} E_j R_j - \sum_{j=n_2+1}^{p_3} E_j - \sum_{j=1}^{q_1} B_j > 0 \tag{1.8}$$

$$|\arg x| < \frac{1}{2} \Omega \pi, \ |\arg y| < \frac{1}{2} \Lambda \pi \tag{1.9}$$

The behavior of the $\overline{H}$ -function of two variables for small values of $|z|$ follows as:

$$\overline{H}[x,y] = 0(|x|^{\alpha}|y|^{\beta}), \max\{|x|,|y|\} \to 0 \tag{1.10}$$

Where

$$\alpha = \min_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{d_j}{\delta_j}\right)\right] \qquad \beta = \min_{1\le j\le m_2}\left[\operatorname{Re}\left(\frac{f_j}{F_j}\right)\right] \tag{1.11}$$

For large value of $|z|$,

$$\overline{H}[x,y] = 0\{|x|^{\alpha'},|y|^{\beta'}\}, \min\{|x|,|y|\} \to 0 \tag{1.12}$$

Where

$$\alpha' = \max_{1\le j\le n_2}\operatorname{Re}\left(K_j\frac{c_j-1}{\gamma_j}\right),\ \beta' = \max_{1\le j\le n_3}\operatorname{Re}\left(R_j\frac{e_j-1}{E_j}\right) \tag{1.13}$$

Provided that $U < 0$ and $V < 0$.

If we take $K_j = 1(j = 1,2,\ldots,n_2), L_j = 1(j = m_2+1,\ldots,q_2), R_j = 1(j = 1,2,\ldots,n_3), S_j = 1(j = m_3+1,\ldots,q_3)$ in (2.1), the $\overline{H}$ -function of two variables reduces to H -function of two variables due to [9].

If we set $n_1 = p_1 = q_1 = 0$, the $\overline{H}$ -function of two variables breaks up into a product of two $\overline{H}$ -function of one variable namely

$$\overline{H}{}^{0,0:m_2,n_2:m_3,n_3}_{0,0:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}--:(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n+1,p_2}:(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\--:(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}:(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]$$

$$= \overline{H}{}^{m_2,n_2}_{p_2,q_2}\left[x\middle|\begin{matrix}(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}\\(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}\end{matrix}\right]\overline{H}{}^{m_3,n_3}_{p_3,q_3}\left[y\middle|\begin{matrix}(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right] \tag{1.14}$$

If $\lambda > 0$, we then obtain

$$\lambda^2\,\overline{H}{}^{0,n_1:\,m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x^{\lambda}\\y^{\lambda}\end{matrix}\middle|\begin{matrix}(a_j,\lambda\alpha_j;A_j)_{1,p_1},(c_j,\lambda\gamma_j;K_j)_{1,n_2},(c_j,\lambda\gamma_j)_{n_2+1,p_2},(e_j,\lambda E_j;R_j)_{1,n_3},(e_j,\lambda E_j)_{n_3+1,p_3}\\(b_j,\lambda\beta_j;B_j)_{1,q_1},(d_j,\lambda\delta_j)_{1,m_2},(d_j,\lambda\delta_j;L_j)_{m_2+1,q_2},(f_j,\lambda F_j)_{1,m_3},(f_j,\lambda F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]$$

$$= \overline{H}{}^{0,n_1:\,m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right] \tag{1.15}$$

$$\overline{H}{}^{0,n_1:\,m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}1/x\\1/y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right]$$

$$= \overline{H}{}^{0,n_1:\,m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(1-b_j,\beta_j;B_j)_{1,q_1},(1-d_j,\delta_j)_{1,m_2},(1-d_j,\delta_j;L_j)_{m_2+1,q_2},(1-f_j,F_j)_{1,m_3},(1-f_j,F_j;S_j)_{m_3+1,q_3}\\(1-a_j,\alpha_j;A_j)_{1,p_1},(1-c_j,\gamma_j;K_j)_{1,n_2},(1-c_j,\gamma_j)_{n_2+1,p_2},(1-e_j,E_j;R_j)_{1,n_3},(1-e_j,E_j)_{n_3+1,p_3}\end{matrix}\right] \tag{1.16}$$

II. MAIN RESULTS

If t be an arbitrary parameter and α', α'' be positive real numbers, then it can be verified that

$$D''\left\{\overline{H}\left[z_1 t^{\alpha'}, z_2 t^{\alpha''}\right]\right. = t^{e-3}\,\overline{H}^{o,n_1+1:\ m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}$$

$$\left[\begin{matrix} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{matrix}\left|\begin{matrix} (1-e+n;\alpha',\alpha''),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(1-e;\alpha',\alpha''),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right] \quad (2.1)$$

And

$$t^{e-1}\,\overline{H}^{0,0:m_2,n_2:m_3,n_3}_{0,0:p_2,q_2:p_3,q_3}\left[\begin{matrix} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{matrix}\left|\begin{matrix} --:(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n+1,p_2}:(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ --:(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}:(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right] =$$

$$t^{\frac{(e-1)}{2}}\,\overline{H}^{m_2,n_2}_{p_2,q_2}\left[z_1 t^{\alpha'}\left|\begin{matrix} (c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2} \\ (d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2} \end{matrix}\right.\right] \times t^{\frac{(e-1)}{2}}\,\overline{H}^{m_3,n_3}_{p_3,q_3}\left[z_2 t^{\alpha''}\left|\begin{matrix} (e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right] \quad (2.2)$$

Differentiating (2.2) two times w.r.t. t and simplifying, it follows by induction that

$$\overline{H}^{0,1:m_2,n_2:m_3,n_3}_{1,1:p_2,q_2:p_3,q_3}\left[\begin{matrix} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{matrix}\left|\begin{matrix} (1-e+n;\alpha',\alpha''):(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n+1,p_2}:(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (1-e;\alpha',\alpha''):(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2}:(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix}\right.\right]$$

$$= \sum_{\substack{n_1,n_2=0 \\ n_1+n_2=n}}^{n} \frac{n!}{n_1!\,n_2!}\,\overline{H}^{0,0:m_2,n_2+1:m_3,n_3+1}_{0,0:p_2+1,q_2+1:p_3+1,q_3+1}$$

$$\left[\begin{matrix} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{matrix}\left|\begin{matrix} --:\left(1+n_1-\frac{e+1}{2},\alpha';1\right)(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n+1,p_2}:\left(1+n_2-\frac{e+1}{2},\alpha'';1\right)(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ --:(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{e+1}{2},\alpha';1\right):(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3},\left(1-\frac{e+1}{2},\alpha'';1\right) \end{matrix}\right.\right] \quad (2.3)$$

(2.3)readily admits an extension and we have

$$= \sum_{\substack{n_1,n_2=0 \\ n_1+n_2=n}}^{n} \frac{n!}{n_1!\,n_2!}\,\overline{H}^{o,n_1:\ m_2,n_2+1:\ m_3,n_2+1}_{p_1,q_1:p_2+1,q_2+1;p_2+1,q_2+1}$$

$$\left[\begin{matrix} z_1 t^{\alpha'} \\ z_2 t^{\alpha''} \end{matrix}\left|\begin{matrix} (a_j,\alpha_j;A_j)_{1,p_1},\left(1+n_1-\frac{e+1}{2},\alpha';1\right),(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},\left(1+n_2-\frac{e+1}{2},\alpha'';1\right),(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{e+1}{2},\alpha';1\right),(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3},\left(1-\frac{e+1}{2},\alpha'';1\right) \end{matrix}\right.\right] \quad (2.4)$$

Considering various other forms that (2.1)admits, similar other results can be obtained.

In the next place, in view of (2.1) we note that the 2-dimensional Mellin-transformation ([6],11.2) M'' of the $\overline{H}$ -function of two variables is given by

$$M''(\overline{H}) = Q(-\xi, -\eta)$$

Provided

$$-\min_{1\le j\le m_2} \operatorname{Re}\left(\frac{d_j}{\delta_j}\right) < \xi < \max_{1\le j\le n_2} \operatorname{Re}\left(K_j\,\frac{c_j-1}{\gamma_j}\right)$$

$$-\min_{1\le j\le m_3} \operatorname{Re}\left(\frac{f_j}{F_j}\right) < \eta < \max_{1\le j\le n_3} \operatorname{Re}\left(R_j\,\frac{e_j-1}{E_j}\right)$$

We also note that, since (1.7 (30) of Erdelyi [7]) for a positive integer N ,

$$\psi(a+N)-\psi(a)=\sum_{k=1}^{N}\frac{(-1)^{k-1}N!}{k(N-k)!}\frac{\psi(a)}{\psi(a+k)};\psi(a)=\frac{\Gamma'(a)}{\Gamma(a)},$$

Partial differentiation of the gamma product $\Gamma\left(1-\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\Gamma\left(1+\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)$ w.r.t. the arbitrary parameter e at $e=N$ can be expressed as a finite sum

$$\frac{\partial}{\partial e}\left\{\Gamma\left(1-\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\Gamma\left(1+\frac{e}{2}+\alpha'\xi+\alpha''\eta\right)\right\}\Bigg|_{e=N}$$

$$=\frac{1}{2}\Gamma\left(1-\frac{N}{2}+\alpha'\xi+\alpha''\eta\right)\sum_{k=1}^{N}\frac{(-1)^{k-1}N!}{k(N-k)!}\frac{\Gamma\left(1+\frac{N}{2}+\alpha'\xi+\alpha''\eta\right)}{\Gamma\left(1+\frac{N}{2}-k+\alpha'\xi+\alpha''\eta\right)},$$

Where α',α'' are positive real numbers.

Thus for $n>0, N>0$, we have

$$M''\left\{\overline{H}^{\,o,n_1+2:\;m_2,n_2:m_3,n_2}_{\,p_1+2,q_1:p_2,q_2;p_2,q_2}\left[\begin{matrix} z_1\\ z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{e}{2};\alpha',\alpha''\right),\left(\frac{e}{2};\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},\left(1+n_2-\frac{e+1}{2},\alpha'';1\right),(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},\left(1-\frac{e+1}{2},\alpha';1\right),(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3},\left(1-\frac{e+1}{2},\alpha'';1\right)\end{matrix}\right.\right]_{e=N}\right\}$$

$$=\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}N!}{k(N-k)!}\frac{\Gamma\left(1+\frac{N}{2}-\alpha'\xi-\alpha''\eta\right)\Gamma\left(1-\frac{N}{2}-\alpha'\xi-\alpha''\eta\right)}{\Gamma\left(1+\frac{N}{2}-k-\alpha'\xi-\alpha''\eta\right)}$$

$$\times Q(-\xi,-\eta)$$

$$M''\left\{\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k(N-k)!}\overline{H}^{\,o,n_1+3:\;m_2,n_2:m_3,n_2}_{\,p_1+3,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{matrix} z_1\\ z_2\end{matrix}\right.\right.$$

$$\left.\left.\left|\begin{matrix}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2};\alpha',\alpha''\right),(a_j,\alpha_j;A_j)_{1,p_1},,(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3}\\ (b_j,\beta_j;B_j)_{1,q_1},\left(-\frac{N}{2}+k;\alpha',\alpha''\right),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3}\end{matrix}\right.\right]\right\} \quad (2.5)$$

But for $z^{(i)}=u^{-\alpha^{(i)}}, i=1,2$, (2.5) can be written as

$$\frac{\partial}{\partial e}\left\{\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\right.\left|\begin{array}{l}\left(-\frac{e}{2};\alpha',\alpha''\right),\left(\frac{e}{2};\alpha',\alpha''\right),\left(a_j,\alpha_j;A_j\right)_{1,p_1},\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(c_j,\gamma_j\right)_{n_2+1,p_2},\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3} \\ \left(b_j,\beta_j;B_j\right)_{1,q_1},\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2},\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{array}\right]_{e=N}\right\}$$

$$=\frac{N!}{2}\sum_{k=1}^{N}\frac{(-1)^N u^{\frac{N}{2}}}{k(N-k)!}$$

$$D_u^{N-k}\left\{u^{\frac{N}{2}-k}\,\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\right.\left|\begin{array}{l}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2};\alpha',\alpha''\right),\left(a_j,\alpha_j;A_j\right)_{1,p_1},\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(c_j,\gamma_j\right)_{n_2+1,p_2},\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3} \\ \left(b_j,\beta_j;B_j\right)_{1,q_1},\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2},\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{array}\right]\right\} \tag{2.6}$$

If we express the derivative into a sum, carry out the differentiations, interchange the order of summation and simplify, we obtain

$$\frac{\partial}{\partial e}\left\{\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\right.\left|\begin{array}{l}\left(-\frac{e}{2};\alpha',\alpha''\right),\left(\frac{e}{2};\alpha',\alpha''\right),\left(a_j,\alpha_j;A_j\right)_{1,p_1},\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(c_j,\gamma_j\right)_{n_2+1,p_2},\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3} \\ \left(b_j,\beta_j;B_j\right)_{1,q_1},\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2},\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{array}\right]_{e=N}\right\}$$

$$=\frac{N!}{2}\sum_{p=0}^{N-1}\frac{(-1)}{p!(N-p)!}\overline{H}^{o,n_1+2:\ m_2,n_2:m_3,n_2}_{p_1+2,q_1:p_2,q_2;p_2,q_2}$$

$$\left[\begin{array}{c} z_1 \\ z_2 \end{array}\right.\left|\begin{array}{l}\left(-\frac{N}{2};\alpha',\alpha''\right),\left(\frac{N}{2}+p;\alpha',\alpha''\right),\left(a_j,\alpha_j;A_j\right)_{1,p_1},\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(c_j,\gamma_j\right)_{n_2+1,p_2},\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3} \\ \left(b_j,\beta_j;B_j\right)_{1,q_1},\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2},\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{array}\right]\right\} \tag{2.7}$$

Similar other results can be obtained by considering products or quotients of such gamma functions whose partial derivatives w.r.t. the arbitrary parameter involved can be expressed as a finite sum.

For example, for the quotient

$$\frac{\Gamma(1-e-N+\alpha'\xi+\alpha''\eta)}{\Gamma(1-e+\alpha'\xi+\alpha''\eta)},$$

We have

$$\frac{\partial}{\partial e}\left\{\overline{H}^{o,n_1+1:\ m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\right.\left|\begin{array}{l}(e+N;\alpha',\alpha''),\left(\frac{e}{2};\alpha',\alpha''\right),\left(a_j,\alpha_j;A_j\right)_{1,p_1},\left(c_j,\gamma_j;K_j\right)_{1,n_2},\left(c_j,\gamma_j\right)_{n_2+1,p_2},\left(e_j,E_j;R_j\right)_{1,n_3},\left(e_j,E_j\right)_{n_3+1,p_3} \\ \left(b_j,\beta_j;B_j\right)_{1,q_1},(e;\alpha',\alpha''),\left(d_j,\delta_j\right)_{1,m_2},\left(d_j,\delta_j;L_j\right)_{m_2+1,q_2},\left(f_j,F_j\right)_{1,m_3},\left(f_j,F_j;S_j\right)_{m_3+1,q_3}\end{array}\right]\right\}$$

$$= N!\sum_{k=0}^{N} \frac{(-1)^{k-1}}{p(N-k)!} \overline{H}^{o,n_1+1:\ m_2,n_2:m_3,n_2}_{p_1+1,q_1+1:p_2,q_2;p_2,q_2}$$

$$\left[\begin{matrix} z_1 \\ z_2 \end{matrix} \left| \begin{matrix} (e+N;\alpha',\alpha''),(a_j,\alpha_j;A_j)_{1,p_1},(c_j,\gamma_j;K_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},(e_j,E_j;R_j)_{1,n_3},(e_j,E_j)_{n_3+1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(e+k;\alpha',\alpha''),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;L_j)_{m_2+1,q_2},(f_j,F_j)_{1,m_3},(f_j,F_j;S_j)_{m_3+1,q_3} \end{matrix} \right. \right]\Bigg\} \quad (2.8)$$

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