A Minimum Quadratic Unbiased Estimation (Minque) Of Parameters In A Linear Regression Model With Sperical Disturbances

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Abstract:- The present study considered the familiar Gauss-Markov linear model Y = X β + ∈ in which the error vector ∈ has a zero mean vector and a covariance matrix ϕ, a diagonal matrix whose ith element is σ_i^2, the variance of i[th] observation Y_i. Rao (1970) has suggested that the MINQUE theory for the vector of these heteroscedastic variances. In the present work, it has been assumed that the variance of error term will be a linear combination of certain independent variables i.e., $\sigma_i^2 = Z_i^1$ α. Under this assumption the heteroscedastic variances and the parameters of the linear model have hence been estimated by using MINQUE theory.

Index terms: Heteroscedasticity, Homoscedastic, Minimum Quadratic Unbiased Estimation **(**MINQUE**)**

1 INTRODUCTION

In most of the economic applications of the linear regression model. It is found difficult to justify the assumption of homoscedastic disturbances. Failure to correct the model, heteroscedasticity produces inefficient coefficient estimates and invalid inferences concerning their true underlying values. Some useful evidence on the extent of the inefficiency has been reported in an article by Geary (1966) and a book by Goldfeld and Quandt (1972). For this reason a number of proposals have been made that seek to eliminate any heteroscedasticity each generally requires the specification of some particular functional form for the heteroscedasticity. Such knowledge appears to be presumptions. A large spectrum of estimation procedures has been developed, with a view to quicken the interest since the last few years. Approaches to the solutions of this problem contain the overall maximum likelihood estimation, weighted least squares estimation and the transformation of variables.

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The main contribution relating to the estimation for linear models under the problem of heteroscedastcity, has been made by Park (1966), Rao (1970, 1971), Theil (1971), Goldfeld and Quandt 91972), Hartley and Jayatillake (1973), Horn, Horn and Duncan (1975), Harvey (1976), Ameiya (1973, 1977), Taylor (1978), Magnus (1978), Jobson and Fuller (1980) and Lahiri and Egy (1981). In the present research work, an attempt had been made to estimate the heteroscedastic linear regression model in the similar lines of MINQUE theory given by Rao (1970).

2. SPECIFICATION OF A LINEAR REGRESSION MODEL WITH UNEQUAL VARIANCES

Consider a classical linear model

$$Y_{nx1} = X_{nxk}\, \beta_{kx1} + \in_{nx1} \qquad \text{… (2.1)}$$

With the following heteroscedastic assumptions on disturbances :

$$E[\in] = 0$$

and $$E[\in \in^{I}] = \phi \text{ or } \sigma^2 \psi$$

Where, $\phi = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2,)$ is the dispersion matrix of error vector ∈

The OLS estimator $\hat{\beta} = (X^I X)^{-I} (X^I Y)$ is an unbiased estimator of β and its covariance matrix is given by

$$\text{Var}(\hat{\beta}) = (X^I X)^{-I} (X^I \phi X) (X^I X)^{-I} \qquad \text{… (2.2.)}$$

i.e., The optimal minimum variance property of OLS estimator no longer holds. If ϕ is symmetric and positive definite, then a non-singular matrix T_{nxn} can be found such that

$$\phi = TT^{\prime} \quad \ldots (2.3)$$

$$\Rightarrow \quad T^{-1} \phi T^{\prime -1} = I \quad \text{or}$$

$$(T^{\prime} \phi^{-1} T)^{-1} = I$$

Pre multiplying the model (2.1) by T^{-1} that gives

$$T^{-1} Y = (T^{-1} X) \beta + T^{-1} \epsilon$$

or $$Y^* = X^* \beta + \epsilon \quad \ldots (2.4)$$

Where $Y^* = T^{-1} Y$, $X^* = T^{-1} X$, $\epsilon^* = T^{-1} \epsilon$

$$\therefore \quad E(\epsilon^*) = E(T^{-1} \epsilon) = T^{-1} E(\epsilon) = 0$$

and $$E(\epsilon^* \epsilon^{*1}) = E[(T^{-1} \epsilon)(T^{-1} \epsilon)^{\prime}]$$

$$= (T^{-1}) E(\epsilon \epsilon^{\prime}) (T^{-1})^{\prime}$$

$$= T^{-1} \phi T^{\prime -1}$$

$$\Rightarrow E(\epsilon^* \epsilon^{*\prime}) = (T^{-1} \phi T)^{-1} \quad \ldots (2.5)$$

Thus, the application of OLS estimation to be transformed model $Y^* + X^* \beta + \epsilon^*$ gives the estimates that posses all the optimal properties. The estimator for β is now given by

$$\tilde{\beta} = (X^{*\prime} X^*)^{-1} (X^{*\prime} Y^*)$$

$$= [(T^{-1} X)^{\prime} (T^{-1} X)]^{-1} [(T^{-1} X)^{\prime} (T^{-1} Y)]$$

$$= [X^{\prime} (T^{\prime -1} T^{-1}) X]^{-1} [X^{\prime} (T^{\prime -1} T^{-1}) Y]$$

$$= [X^{\prime} (TT^{\prime})^{-1} X]^{-1} [X^{\prime} (TT^{\prime})^{-1} Y]$$

$$\tilde{\beta} = [X^{\prime} \phi^{-1} X]^{-1} [X^{\prime} \phi^{-1} Y] \quad \ldots (2.6)$$

$$\text{Also, } Var(\tilde{\beta}) = [X^{\prime} \phi^{-1} X]^{-1} \quad \ldots (2.7)$$

In practice, the elements of matrix ϕ are unknown and they may be function of one or more unknown parameters. It is possible to obtain these parameters and hence ϕ is estimated. Then the estimated generalized least squares (EGLS) estimator may be written

$$\hat{\tilde{\beta}} = [X^{\prime} \hat{\phi}^{-1} X]^{-1} [X^{\prime} \hat{\phi}^{-1} Y] \quad \ldots (2.8)$$

3. MINIMUM QUADRATIC UNBIASED ESTIMATION (MINQUE) OF PARAMETERS OF AN HETEROSCEDASTIC LINEAR REGRESSION MODEL

Consider the heteroscedastic structure of Goldfeld and Quandt (1972) for σ_j^2 as

$$\sigma_j^2 = Z_j^{\prime} \alpha, \qquad j = 1, 2, \ldots, n \quad \ldots (3.1)$$

Which implies that the variance of the error term is a linear combination of certain independent variables. Here, Z_j is a (k x 1) vector of fixed known coefficients α is a (kx1) vector of unknown parameters. Now, the problem is to estimate unknown parameters (α's).

Writing the OLS residual vector as :

$$e = Y - X \hat{\beta} \quad \ldots (3.2)$$

$$= X\beta + \epsilon - X(X^{\prime}X)^{-1} X^{\prime} (X\beta + \epsilon)$$

$$= \epsilon - X(X^{\prime}X)^{-1} X^{\prime} \epsilon$$

$$= [I - X(X^{\prime}X)^{-1} X^{\prime}] \epsilon$$

$$\Rightarrow \quad e = M \epsilon \quad \ldots (3.3.)$$

Where $M = [I - (X^{\prime}X)^{-1} X^{\prime}]$ is a symmetric idenpotent matrix of order n.

The covariance matrix of e is given by

$$Var(e) = E[ee^{\prime}] \qquad \because E(\epsilon) = 0$$

$$= E[M \epsilon \epsilon^{\prime} M^{\prime}]$$

$$= M E[\epsilon \epsilon^{\prime}] M^{\prime}$$

$$\Rightarrow \quad Var(e) = M \phi M^{\prime} \quad \ldots (3.4)$$

If m_i stands for the i^{th} row of matrix M, then equation (3.3) may be expressed as

$$e_i = m_i \epsilon, \qquad i = 1, 2, \ldots, n \quad \ldots (3.5)$$

The covariance between e_i and e_j is defined as

$$E[e_i e_j] = m_i \phi m_j^{\prime}, \qquad i \neq j = 1, 2, \ldots, n \quad \ldots (3.6)$$

The estimation equations may be written by dropping the expectation operation in (3.6) as

$e_i\, e_j = m_i\, \phi\, m_j^{|}$

$\Rightarrow e_i^2 = m_i\, \phi\, m_j^{|}$ for i = j

$$\Rightarrow e_i^2 = \sum_{j=1}^{n} m_{ij}\ \sigma_j^2, \quad i = 1, 2, \ldots, n \qquad \ldots (3.7)$$

Here, $\phi = \text{diag}\,(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)$

Equations (3.1) and (3.7) imply that

$$\begin{bmatrix} e_1^2 \\ e_2^2 \\ \cdot \\ \cdot \\ \cdot \\ e_n^2 \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{12}^2 & \cdot & \cdot & \cdot & m_{1n}^2 \\ m_{21}^2 & m_{22}^2 & \cdot & \cdot & \cdot & m_{2n}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{n1}^2 & m_{n2}^2 & \cdot & \cdot & \cdot & m_{nn}^2 \end{bmatrix} \begin{bmatrix} z_1^{|} \\ z_2^{|} \\ \cdot \\ \cdot \\ \cdot \\ z_n^{|} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}$$

$$\Rightarrow \quad \dot{e} = \dot{M}\, Z\, \alpha \qquad \ldots (3.8)$$

Where $\dot{e}$: (nx1) vector of the squares of OLS residuals $\dot{M}$: (nxn) matrix obtained by squaring each element of a symmetric idempotent matrix M. Z : (nxk) matrix of known coefficients on certain independent variables, which reveal the heteroscedasticity α : (kx1) vector of unknown parameters relates to the heteroscedasticity Equation (3.8) may be written as

$$\dot{e} = H\,\alpha \qquad \ldots (3.9)$$

Where $H = \dot{M}\, Z$

Using the MINQUE procedure, an unbiased estimator of α is given by

$$\hat{\alpha} = H^{g}\, \dot{e} \qquad \ldots (3.10)$$

Where H^g is a generalized inverse of H, having an order (kxn) Thus an unbiased estimator of σ_j^2 is given by

$$\hat{\sigma}_j^2 = z_j^{|}\, \hat{\alpha}, \quad j = 1, 2, \ldots, n \qquad \ldots (3.11)$$

or $\hat{\dot{\sigma}} = Z\, \hat{\alpha}$

$$\text{i.e.,} \quad \hat{\dot{\sigma}} = Z\, H^{g}\, \dot{e} \qquad \ldots (3.12)$$

The covariance matrix of $\hat{\dot{\sigma}}$ is given by

$$\text{Var}\,(\hat{\dot{\sigma}}) = z\, \text{Var}\,(\hat{\alpha})\, Z^{|} \qquad \ldots (3.13)$$

Where $\text{Var}\,(\hat{\alpha}) = H^{g}\, \text{Var}\,(\dot{e})\, H^{g|}$

$= (\dot{M}\, Z^{g})\, \text{Var}\,(\dot{e})\, (\dot{M}\, Z^{g|})$

$$\therefore \text{Var}\,(\hat{\dot{\sigma}}) = z\, (\dot{M}\, Z)^{g}\ \text{Var}\,(\dot{e})\ (\dot{M}\, Z)^{g^{|}}\, Z^{|} \qquad \ldots (3.14)$$

The (i, j)th element of Var ($\dot{e}$) is given by

$$\text{Cov}\,(e_i^2, e_j^2) = 2 \left[\sum_{q=1}^{n} m_{iq}\ m_{jq}\ \sigma_q^2 \right] \qquad \ldots (3.15)$$

Now, the proposed estimator for β is given by

$$\beta^* = \left(X^{|}\, \phi^{*-|}\, X \right)^{-|} \left(X^{|}\, \phi^{*-|}\, Y \right) \qquad \ldots (3.16)$$

$$\text{with} \quad \text{Var}\,(\beta^*) = \left(X^{|}\, \phi^{*-|}\, X \right)^{-|} \qquad \ldots (3.17)$$

Where $\phi^* = \text{diag}\,[Z_1^{|}\, \hat{\alpha}, Z_2^{|}\, \hat{\alpha}, \ldots\, Z_n^{|}\, \hat{\alpha}]$

Remarks :

1. One may use the Moore and Penrose inverse matrix H^p in the place of H^g to get an unique solution for α. Here, H^p satisfies the following four conditions :
   - (i) $H\, H^p\, H = H$
   - (ii) $H^p\, H\, H^p = H^p$
   - (iii) $(H^p\, H)^{|} = H^p\, H$
   - (iv) $(H\, H^p)^{|} = H\, H^p$

2. If all the variances are not different, then one may proceed like in the method of Rao (1970).

4. CONCLUSIONS

In the present study, the classical linear regression model with heteroscedastic disturbances has been estimated in the framework of the MINQUE theory suggested by Rao (1970). This research work has considered the familier Gauss-Markov linear model Y = X β + $\in$ in which the error vector $\in$ has a zero mean vector and a covariance matrix ϕ, a diagonal matrix whose i^{th} diagonal element is σ_i^2, the variance of i^{th} observation y_i. It has been assumed that the variance of error term will be a linear combination of certain independent variables i.e., $\sigma_i^2 = Z_i^{|}\alpha$. By incorporating this assumption, the heteroscedastic variances and the parameters of the linear model have hence been estimated by using MINQUE theory.

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